INTRODUCTION TO THE EMERTON–GEE STACK

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1. Overview

1.1. **Introduction.** Let p be a fixed prime number and let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Write $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ equipped with its profinite topology. Fix a prime ℓ (possibly equal to p), an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ and an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$, which induces an embedding $G_{\mathbb{Q}_\ell} \hookrightarrow G_{\mathbb{Q}}$.

Important objects of study in modern algebraic number theory are continuous (almost everywhere unramified) representations

$$G_{\mathbb{Q}} \to \mathrm{GL}_d(\mathbb{Q}_p),$$

for example the ones associated to elliptic curves and modular forms (with $n = 2$) or more generally those found in the étale cohomology of smooth projective varieties over $\mathbb{Q}$. These representations are often studied by restricting them to the decomposition group $G_{\mathbb{Q}_\ell} \subset G_{\mathbb{Q}}$ for some prime ℓ as above.

If we fix a continuous irreducible representation

$$\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_d(\mathbb{F}_p),$$

then it is very interesting to study if and how it lifts to representations ρ with values in $\mathrm{GL}_d(\mathbb{Z}_p)$. For example if $\overline{\rho}$ comes from a modular form one might ask if all lifts ρ come from modular forms. The key technical object used to study such lifting problems is a *universal deformation ring.*

Fix a finite set of primes S of $\mathbb{Q}$ containing p such that $\overline{\rho}$ is unramified outside of S. Then there is a complete Noetherian local ring $R^{\square}_{\overline{\rho},S}$ together with a continuous homomorphism

$$\rho^{\mathrm{univ}}_S : G_{\mathbb{Q}} \to \mathrm{GL}_d(R^{\square}_{\overline{\rho},S}),$$

such that any continuous homomorphism

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_d(A),$$

where A is an Artin local ring with residue field $\mathbb{F}_p$, which is unramified outside of S comes from ρ^{univ}_S via a map $R^{\square}_{\overline{\rho},S} \to A$.

For $\ell \in S$ (including $\ell = p$), there are similar deformation rings $R^{\square}_{\overline{\rho},\ell}$ and continuous homomorphisms

$$\rho^{\mathrm{univ}}_\ell : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_d(R^{\square}_{\overline{\rho},\ell}),$$

which are universal for continuous homomorphisms $G_{\mathbb{Q}_\ell} \to \mathrm{GL}_d(A)$ where A is an Artin local ring with residue field $\mathbb{F}_p$. In particular there are restriction maps

$$R^{\square}_{\overline{\rho},\ell} \to R^{\square}_{\overline{\rho},S}.$$

1.2. **Moduli spaces.** The complete Noetherian local rings discussed above can be thought of as the complete local rings of the 'moduli space of' continuous homomorphisms $G_{\mathbb{Q}_\ell} \to \mathrm{GL}_d$. When $\ell \neq p$ we defined such moduli spaces in the spring quarter, and saw that it was very interesting to study them. For example because they can be used to state a version of the local Langlands correspondence in families (and because Fargues–Scholze recently proved results in this direction).

A slightly fancier perspective is to instead consider the 'moduli stack' of continuous representations $G_{\mathbb{Q}_\ell} \to \mathrm{GL}_d$, which is 'just' the quotient of the 'moduli space' above by the natural conjugation action of GL_d. The goal of this learning seminar is to study the $\ell = p$ version of these moduli stacks. These were introduced a few years ago by Emerton–Gee and are now often referred to as *the Emerton–Gee stack*

Very recently, it has been realised by various people that the resulting stacks should have an important role to play in the mod p^n local Langlands correspondence for GL_d over $\mathbb{Q}_p$. Perhaps we could discuss these conjectures in the next quarter.

1.3. **Universal deformation rings.** From now on we work in the $\ell = p$ sitation and drop ℓ from the notation. Recall that we have fixed an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and that $\mathrm{Gal}_{\mathbb{Q}_p}$ denotes the absolute Galois group of $\mathbb{Q}_p$. Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field $k = \mathcal{O}/\varpi$. Suppose that we are given a continuous

$$\overline{\rho} : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_d(\mathbb{F}_q).$$

Let $\mathrm{Art}_{\mathcal{O}}$ be the category of Artin local $\mathcal{O}$-algebras with residue field k and homomorphisms given by local homomorphisms. There is a functor

$$\mathrm{Art}_{\mathcal{O}} \to \mathrm{Sets}$$

sending R to the set of homomorphisms

$$\rho : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_d(R)$$

that are equal to $\overline{\rho}$ under the canonical isomorphism $R/\mathfrak{m} = k$. This functor is pro-representable by a complete Noetherian local $\mathbb{Z}_q$-algebra $R^{\square}_{\overline{\rho},p} = R^{\square}_{\overline{\rho}}$. It is a recent result of Böckle-Iyengar-Paškūnas that $R^{\square}_{\overline{\rho}}$ is a flat complete intersection of relative dimension $2d^2$ over $\mathcal{O}$. We would like to build a (formal) moduli stack whose versal rings are given by the $R^{\square}_{\overline{\rho}}$.

1.4. **Naive stack of Galois representations.** Let $\mathrm{Nilp}_{\mathbb{Z}_p}$ be the category of $\mathbb{Z}_p$-algebras in which p is nilpotent. Then as a first approximation we can consider the stack $\mathcal{X}^{\mathrm{Gal}}$ on $\mathrm{Nilp}_{\mathbb{Z}_p}$ whose R-valued points is given by the groupoids of continuous Galois representation

$$\rho : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_d(R).$$

Here we consider the discrete topology on $\mathrm{GL}_d(R)$, and morphisms $\rho \to \rho'$ are given by $g \in \mathrm{GL}_d(R)$ such that $g\rho g^{-1} = \rho'$. This stack was considered in [2] and it is proved there that it is a formal algebraic stack.[1] Moreover the natural maps $\mathrm{Spf}\, R^{\square}_{\overline{\rho}} \to \mathcal{X}^{\mathrm{Gal}}$ are versal rings (here we should take $\mathcal{O} = \mathbb{Z}_q$).

Unfortunately, this stack has its drawbacks. Let D be a continuous *semisimple* Galois representation $D : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_d(\overline{\mathbb{F}}_p)$. Then there is a closed substack $\mathcal{X}^{\mathrm{Gal}}_D$, whose R-points consists of those morphisms $\rho : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_d(R)$ such that for all maps $R \to \overline{\mathbb{F}}_p$ the induced Galois representation valued in $\overline{\mathbb{F}}_p$ has semi-simplification equal to D. Then there is an isomorphism

$$\left(\mathcal{X}^{\mathrm{Gal}}\right)_{W(\overline{\mathbb{F}}_p)} \simeq \coprod_D \mathcal{X}^{\mathrm{Gal}}_D.$$

[1]This just means that it is a reasonable geometric object. There will be a talk on algebraic stacks and a talk on formal algebraic stacks to make this precise.

In particular the stack $\mathcal{X}^{\mathrm{Gal}}$ has infinitely many connected components. When $d = 1$, the stack $\mathcal{X}^{\mathrm{Gal}}$ is more or less the disjoint union of all the formal spectra of the formal deformation rings (quotiented by the trivial action of $\mathbb{G}_m$).

1.4.1. The main result of [1] is the construction of a flat Noetherian formal algebraic stack $\mathcal{X}$ over $\mathbb{Z}_p$ containing

$$\mathcal{X}^{\mathrm{Gal}} \subset \mathcal{X}$$

as a closed substack. Moreover, the underlying reduced substack of $\mathcal{X}$ is a Noetherian algebraic stack over $\overline{\mathbb{F}}_p$, in particular it has finitely many irreducible components. Moreover these irreducible components have a natural representation-theoretic interpretation which is closely relate to Serre weight conjectures. These will *not* be moduli stacks of Galois representations in any way, but rather moduli spaces of étale (φ, Γ)-modules.

1.5. **Étale phi-gamma modules.** We now define the Emerton–Gee stack, as always in the case of Galois representations of $G_{\mathbb{Q}_p}$. Things are more complicated for G_K where K is a finite extension of $\mathbb{Q}_p$.

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$ and let $\epsilon : \Gamma \to \mathbb{Z}_p^\times$ be the cyclotomic character. Let $A \in \mathrm{Nilp}_{\mathbb{Z}_p}$ be of finite type over $\mathbb{Z}_p$, and define

$$\mathbb{A}_A := A[[T]][1/T],$$

equipped with the topological ring structure making $A[[T]]$ an open subring equipped with the T-adic topology. Let Γ act on $\mathbb{A}_A$ via

$$\gamma(1 + T) = (1 + T)^{\epsilon(\gamma)},$$

here we are using the fact that the multiplicative group $1 + TA[[T]]$ is naturally a $\mathbb{Z}_p^\times$-module. Let $\varphi : \mathbb{A}_A$ be the morphisms determined by $\varphi(1 + T) = (1 + T)^p$.

Definition 1.5.1. *A rank d étale (φ, Γ)-module over A is a rank d projective $\mathbb{A}_A$-module M equipped with commuting (continuous) semilinear actions of φ and Γ, such that M is generated by the image of $\phi_M : M \to M$.*

Theorem 1.5.2 (Fontaine)**.** *If A is a finite $\mathbb{Z}_p$-module, then there is an equivalence of categories between the category of rank d (ϕ, Γ)-modules over A and the category of projective A-modules of rank d equipped with a continuous action of $G_{\mathbb{Q}_p}$*

Example 1.5.3. Let $\lambda \in \overline{\mathbb{F}}_p^\times$, then there is a rank one ϕ-Gamma module

$$\mathrm{ur}_\lambda := \mathbb{A}_{\overline{\mathbb{F}}_p} \cdot e$$

with trivial Γ action and $\phi(e) = \lambda \cdot e$. This corresponds to the unramified character $G_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^\times$ sending Frobenius to λ.

Example 1.5.4. Let $A = \mathbb{F}_p[X, X^{-1}]$, then there is a rank one (ϕ, Γ)-module

$$\mathrm{ur}_X := \mathbb{A}_A \cdot e$$

with trivial Γ-action and with $\phi(e) = X \cdot e$. We observe that there is *no* continuous unramified character

$$G_{\mathbb{Q}_p} \to A^\times$$

sending Frobenius to X. This is the first example of an algebraic family of (ϕ, Γ)-modules which does come from an algebraic family of Galois representations. However, this is not a very interesting example as there *is* an $A^\times$ valued representation of the Weil group $W_{\mathbb{Q}_p}$.

Example 1.5.5. Fact I: For $\lambda \neq \lambda' \in \overline{\mathbb{F}}_p^\times$ there is a unique isomorphism class of extensions of Galois representations

$$\begin{pmatrix} \mathrm{ur}_\lambda & * \\ 0 & \mathrm{ur}_{\lambda'}. \end{pmatrix}$$

Fact II: There is a (ϕ, Γ)-module over $A = \overline{\mathbb{F}}_p[X, X^{-1}, Y, Y^{-1}, (X-Y)^{-1}]$ which interpolates these Galois representations, but there is no continuous representation of $G_{\mathbb{Q}_p}$ or $W_{\mathbb{Q}_p}$ on a rank two projective A-module corresponding to it. Proof omitted, but the idea seems to be that as we vary λ and λ' then the ramification of the corresponding extension is unbounded. On the other hand, a continuous representation of $W_{\mathbb{Q}_p}$ on A (which has the discrete topology), has to factor through a finite quotient of the inertia group.

1.6. **The Emerton–Gee stack.** We are now ready to define the Emerton–Gee stack $\mathcal{X}_d$. It will be the[2] functor from $\mathrm{Nilp}_{\mathbb{Z}_p}$ to groupoids which sends a finitely generated $\mathbb{Z}_p$ algebra A where p is nilpotent to the groupoid of rank d étale (ϕ, Γ)-modules over A.

It follows from the theorem of Fontaine mentioned above that the deformation theory of this moduli problem is the same as the deformation theory of Galois representations.

Theorem 1.6.1 (Emerton–Gee)**.** *The functor to groupoids $\mathcal{X}_d$ is a Noetherian formal algebraic stack that is flat over $\mathrm{Spf}\,\mathbb{Z}_p$. Moreover the underlying reduced stack $\mathcal{X}_{d,\mathrm{red}}$ is an algebraic stack over $\mathbb{F}_p$, which is equidimensional of dimension $d(d-1)/2$.*

1.7. **Irreducible components of the underlying reduced stack.** We need the following definition in what follows

Definition 1.7.1. *A* Serre weight *for GL_d over $\mathbb{Q}_p$ is an irreducible representation of $\mathrm{GL}_d(\mathbb{F}_p)$ on a $\overline{\mathbb{F}}_p$-vector space V.*

Isomorphism classes of Serre weights are determined by their highest weight, which is a tuple of integers $\underline{k} = (k_i)_{1 \leq i \leq d}$ such that:

- We have $p - 1 \geq k_i - k_{i+1} \geq 0$ for all $1 \leq i \leq d-1$ and
- we have $p - 1 > k_d \geq 0$.

We will write $V_{\underline{k}}$ for the Serre weight V associated to a highest weight $\underline{k}$ as above.

Example 1.7.2. The isomorphism classes of irreducible representations of $\mathrm{GL}_2(\mathbb{F}_p)$ on $\overline{\mathbb{F}}_p$-vector spaces are given by

$$\mathrm{Sym}^{i-1} V_{\mathrm{std}} \otimes \mathrm{Det}^j V_{\mathrm{std}},$$

where $0 \leq j < p-1$ and $1 \leq i \leq p$. The representation $\mathrm{Sym}^{i-1} V_{\mathrm{std}} \otimes \mathrm{Det}^j$ has highest weight $(i+j-1, j)$.

Remark 1.7.3. I think it is true that the irreducible representations of $\mathrm{GL}_d(\mathbb{F}_p)$ of highest weight $\underline{k}$ arises from the algebraic irreducible representations of $\mathrm{GL}_d(\overline{\mathbb{F}}_p)$ of highest weight $\underline{k}$ via restriction.

Theorem 1.7.4 (Emerton–Gee)**.** *The irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ are in bijection with isomorphism classes of Serre weights. We will write*

$$\mathcal{X}_d^{\underline{k}} \subset \mathcal{X}_{d,\mathrm{red}}$$

for the corresponding closed substack.

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[2] There is a unique limit preserving functor with this property

Example 1.7.5. When $d = 2$, the irreducible component corresponding to $\mathrm{Sym}^{i-1} V_{\mathrm{std}} \otimes \mathrm{Det}^j$ generically parametrises those representations of the form

$$\begin{pmatrix} \mathrm{ur}_{\alpha\beta}\omega^{i+j} & * \\ 0 & \mathrm{ur}_\beta \omega^j \end{pmatrix},$$

where ω denotes the cyclotomic character.

1.8. **Crystalline substacks.** A *Hodge type* is a decreasing sequence of integers $\underline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_d)$. Recall that a *crystalline*[3] continuous representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ has a set of Hodge–Tate weights which gives us a Hodge type after ordering them.

Theorem 1.8.1. *There is a closed substack*

$$\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}} \subset \mathcal{X}_d$$

which is flat over $\mathrm{Spf}\,\mathbb{Z}_p$*, which is characterised by the following property: For any* finite flat $\mathbb{Z}_p$*-algebras* A*, the subcategory*

$$\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}(A) \subset \mathcal{X}_d(A)$$

is precisely the subcategory of those representations that are crystalline with Hodge–Tate weights $\underline{\lambda}$ *after inverting* p*. Moreover, the morphism*

$$\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}} \to \mathrm{Spf}\,\mathbb{Z}_p$$

is representable of relative dimension.

Corollary 1.8.2. *The formal algebraic stack* $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\mathbb{F}_p}$ *is an algebraic stack, and its irreducible components are irreducible components of* $\mathcal{X}_{d,\mathrm{red}}$.

An important and mysterious question is *which* irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ occur in the special fiber of $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$. One would perhaps also like to know the multiplicities with which the irreducible components occur. This problem is closely related to generalisations of the weight part of Serre's conjecture as well as the Breuil–Mézard conjecture.

1.9. **The 'universal' geometric Breuil–Mézard conjecture.** Let $\underline{\lambda}$ be a Hodge type as above. Then we can associate to λ an algebraic representation $M_{\underline{\lambda}}$ of $\mathrm{GL}_d(\mathbb{Z}_p)$, using the theory of highest weight representations. To be precise we want to take the representation of highest weight $\underline{\zeta}$ where $\zeta_i = k_i - (d-i)$.

Let $\overline{\sigma}^{\mathrm{crys}}(\underline{\lambda})$ denote the semi-simplification of the mod p reduction of this representation, which is naturally a representation of $\mathrm{GL}_d(\mathbb{F}_p)$. Then there are unique integers $n_{\underline{k}}^{\mathrm{crys}}(\underline{\lambda})$ such that

$$\overline{\sigma}^{\mathrm{crys}}(\underline{\lambda}) \simeq \bigoplus V_{\underline{k}}^{\oplus n_{\underline{k}}^{\mathrm{crys}}(\underline{\lambda})}.$$

We will consider the free abelian group generated by the irreducible components $\mathcal{X}_d^{\underline{k}} \subset \mathcal{X}_{d,\mathrm{red}}$ of $\mathcal{X}_{d,\mathrm{red}}$. Elements of this free abelian group will be called cycles on $\mathcal{X}_{d,\mathrm{red}}$. Cycles are called *effective* if they are nonnegative linear combinations of irreducible components.

Conjecture 1 (The 'universal' geometric Breuil–Mézard conjecture)**.** *There are cycles* $Z_{\underline{k}}$ *on* $\mathcal{X}_{d,\mathrm{red}}$ *such that for every Hodge type* $\underline{\lambda}$ *there is an equality of cycles*

$$\mathcal{X}_{d,\mathbb{F}_p}^{\mathrm{crys},\underline{\lambda}} = \sum_{\underline{k}} n_{\underline{k}}^{\mathrm{crys}}(\underline{\lambda}) Z_{\underline{k}}.$$

Remark 1.9.1. Trying to formulate a version of this conjecture without the Emerton–Gee stack but instead in terms of the crystalline deformation rings is somewhat tedious.

[3]For example the Tate module of an abelian variety with good reduction

Describing the cycles $Z_{\underline{k}}$ is conjecturally equivalent to generalisations of the weight part of Serre's conjecture. In particular, it is probably very hard. One thing that we expect is that the support of the cycle $Z_{\underline{k}}$ contains the irreducible component $\mathcal{X}_d^{\underline{k}}$, but already for GL_2 there are examples when the support of $Z_{\underline{k}}$ is larger than just $\mathcal{X}_d^{\underline{k}}$ (but generically it doesn't). Moreover, for GL_2 over $\mathbb{Q}_{p^f}$ we expect generic $\underline{k}$ to have $Z_{\underline{k}}$ with support larger than $\mathcal{X}_d^{\underline{k}}$.

The end goal of this quarter's learning seminar is to understand the above conjecture, for example by defining all of the objects involved and proving that some of them have some of the properties that I've claimed above that they have.

References

[1] Matthew Emerton and Toby Gee, *Moduli stacks of étale (phi,Gamma)-modules and the existence of crystalline lifts*, arXiv e-prints (August 2019), available at `1908.07185`.

[2] Carl Wang-Erickson, *Algebraic families of Galois representations and potentially semi-stable pseudodeformation rings*, Math. Ann. **371** (2018), no. 3-4, 1615–1681. MR3831282